

# GEOMETRIC CONSTRUCTION OF CLUSTER ALGEBRAS AND CLUSTER CATEGORIES

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ABSTRACT. In this note we explain how to obtain cluster algebras from triangulations of (punctured) discs following the approach of [FST06]. Furthermore, we give a description of  $m$ -cluster categories via diagonals (arcs) in (punctured) polygons and of  $m$ -cluster categories via powers of translation quivers as given in joint work with R. Marsh ([BM08a], [BM07]).

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## 1. INTRODUCTION

This article is an expanded version of a talk presented at the Courant-Colloquium “Göttingen trends in Mathematics” in October 2007. It is a survey on two approaches to cluster algebras and ( $m$ -)cluster categories via geometric constructions.

Cluster algebras were introduced in 2001 by Fomin and Zelevinsky, cf. [FZ02a]. They arose from the study of two related problems.

**Problem 1** (Canonical basis). Understand the *canonical basis* (Lusztig), or *crystal basis* (Kashiwara) of quantized enveloping algebras associated to a semisimple complex Lie algebra. It is expected that the positive part of the quantized enveloping algebra has a (quantum) cluster algebra structure, with the so-called cluster monomials forming part of the dual canonical basis.

This picture motivated the definition of *cluster variables*.

**Problem 2** (Total positivity). An invertible matrix with real entries is called *totally positive* if all its minors are positive. This notion has been extended to all reductive groups by Lusztig [Lu94]. To check total positivity for an upper uni-triangular matrix, only a certain collection of the non-zero minors needs to be checked (disregarding the minors which are zero because of the uni-triangular form). The minimal sets of such all have the same cardinality. When one of them is removed, it can often be replaced by a unique alternative minor. The two minors are connected through a certain relation.

This exchange (*mutation for minors*) motivated the definition of cluster mutation.

The subject of cluster algebra is a very young and dynamic one. In the past few years, connections to various other fields arose. We briefly mention a few of them here.

- Poisson geometry (integrable systems), Teichmüller spaces (local coordinate systems), cf. Gekhtman-Shapiro-Vainshtein [GSV03, GSV05] and Fock-Goncharov [FG06];
- $Y$ -systems in thermodynamic Bethe Ansatz (families of rational functions defined by recurrences which were introduced by Zamolodchikov [Za91]). Cf. [FZ02a];
- Stasheff polytopes, associahedra, Chapoton-Fomin-Zelevinsky [CFZ02];
- ad-nilpotent ideals of Borel subalgebras in Lie algebras, Panyushev [Pa04];
- Preprojective algebra models, Geiss-Leclerc-Schröer, [GLS05], [GLS07];
- Representation theory, tilting theory, etc., Cf. e.g. [BMRRT05].

In this article, we will first recall triangulations of surfaces with marked points and associate certain integral valued matrices to them. Then we will give a brief introduction to cluster algebras (Section 3). In Section 4 we show how to associate cluster algebras to triangulations of (punctured) discs. Then we explain what cluster categories and  $m$ -cluster categories are (Section 5) and give a combinatorial model to describe  $m$ -cluster categories via arcs in a polygon in Section 6, cf. Theorems 6.3, 6.4 as given in our joint work with R. Marsh ([BM08a], [BM07]). In addition, we obtain a descriptions of the  $m$ -cluster categories using the notion of the power of a translation quiver (Theorem 6.5). At the end we describe connections to other work, pose several questions and show new directions in this young and dynamic field (Section 7).

## 2. TRIANGULATED SURFACES

In this section we recall triangulation of surfaces following the approach of Fomin, Shapiro and Thurston [FST06]. Let  $S$  be a connected oriented Riemann surface with boundary. Fix a finite set  $M$  of *marked points* on  $S$ . Marked points in the interior of  $S$  are called *punctures*.

We consider triangulations of  $S$  whose vertices are at the marked points in  $M$  and whose edges are pairwise non-intersecting curves, so-called *arcs* connecting marked points. The most important example for us is the case where  $S$  is a disc with marked points on the boundary and with at most one puncture. We will later restrict to that case but for the moment we explain the general picture.

It is convenient to exclude cases where there are no such triangulations (or only one such). We always assume that  $M$  is non-empty and that each boundary component has at least one marked point. And we disallow the cases  $(S, M)$  with one boundary component,  $|M| = 1$  with  $\leq 1$  puncture and  $|M| \in \{2, 3\}$  with no puncture.

In case  $S$  is a (punctured) disc we will also call it a *(punctured) polygon*. E.g. if  $(S, M)$  has three marked points on the boundary and a puncture, we will say that  $S$  is a once-punctured triangle.

Note that the pair  $(S, M)$  is defined (up to homeomorphism) by the genus of  $S$ , by the numbers of boundary components, of marked points on each boundary component and of punctures. Two examples of such triangulations are given in Figure 1.

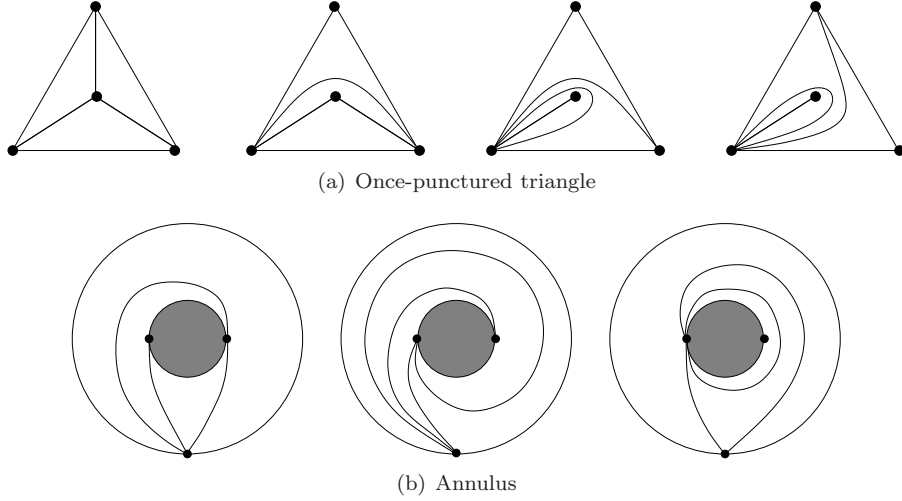


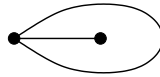
FIGURE 1. Examples of triangulations

**Definition.** A curve in  $S$  (up to isotopy relative  $M$ ) is an *arc*  $\gamma$  in  $(S, M)$  if

- (i) the endpoints of  $\gamma$  are marked points in  $M$ ;
- (ii)  $\gamma$  does not intersect itself (but its endpoints might coincide);
- (iii) relative interior of  $\gamma$  is disjoint from  $M$  and from the boundary of  $S$ ;
- (iv)  $\gamma$  does not cut out an unpunctured monogon or digon.

The set of all arcs in  $(S, M)$  is usually infinite as we can already see in the case of the annulus of Figure 1(b). One can show that it is finite if and only if  $(S, M)$  is a disk with at most one puncture, i.e. if  $(S, M)$  is the object of our interest.

Two arcs are said to be *compatible* if they do not intersect in the interior of  $S$ . An *ideal triangulation* is a maximal collection  $T$  of pairwise compatible arcs. The arcs of  $T$  cut  $S$  into the so-called *ideal triangles*. These triangles may be self-folded, e.g. along the horizontal arc in the picture below:



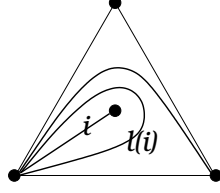
An easy count shows that the once-punctured triangle has ten ideal triangulations, the four of figure 1, with the rotations of the last three (by  $120^\circ$  and  $240^\circ$ ).

In fact we can say more: the number of arcs in an ideal triangulation is an invariant of  $(S, M)$ , we call it the *rank* of  $(S, M)$ . There is a formula for it, cf. [FG07]: if  $g$  is the genus of  $S$ ,  $b$  the number of boundary components,  $p$  the number of punctures,  $c$  the number of marked points on the boundary, then the rank of  $(S, M)$  is

$$6g + 3b + 3p + c - 6$$

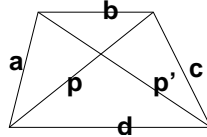
The rank of the once punctured triangle of Figure 1(a) is thus three as expected.

For small rank, [FST06, Example 2.12] gives a list of all possible choices of  $(S, M)$ . The word “type” appearing in the list refers to the Dynkin type of to the corresponding cluster algebra as will be explained later:

FIGURE 2. Enclosing loop  $l(i)$  of the arc  $i$ 

- |        |   |
|--------|---|
| Rank 1 | unpunctured square (type $A_1$ )  |
| Rank 2 | unpunctured pentagon (type $A_2$ )  |
|        | once-punctured digon (type $A_1 \times A_1$ )                             |
|        | annulus with one marked point on each boundary component                  |
| Rank 3 | unpunctured hexagon (type $A_3$ )   |
|        | once-punctured triangle (type $A_3 = D_3$ )                               |
|        | annulus with one marked point on one boundary component, two on the other |
|        | once-punctured torus.   |

If  $T$  is an ideal triangulation of  $(S, M)$  and  $p$  an arc of  $T$  as in the picture below, we can replace  $p$  by an arc  $p'$  through a so-called flip or Whitehead move:



Here we allow that some of the sides  $\{a, b, c, d\}$  coincide. A consequence of a result of Hatcher ([Ha91]) is that for any two ideal triangulations  $T$  and  $T'$  there exists a sequence of flips leading from  $T$  to  $T'$ .

We next want to associate a matrix to an ideal triangulation of  $(S, M)$ . This works as follows. Let  $T$  be an ideal triangulation of  $(S, M)$ , label the arcs of  $T$  by  $1, 2, \dots, n$ . Then define  $B(T)$  to be the following  $n \times n$ -square matrix

$$B(T) = \sum_{\Delta} B^{\Delta}$$

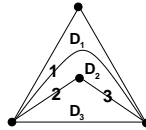
where the  $n \times n$ -matrices  $B^{\Delta}$  are defined for each triangle  $\Delta$  of  $T$  by

$$b_{ij}^{\Delta} = \begin{cases} 1 & \text{if } \Delta \text{ has sides } i \text{ and } j \text{ where } j \text{ is a clockwise neighbour of } i; \\ -1 & \text{if } \Delta \text{ has sides } i \text{ and } j \text{ where } i \text{ is a clockwise neighbour of } j; \\ 0 & \text{otherwise.} \end{cases}$$

The matrix  $B(T)$  is skew-symmetric with entries  $0, \pm 1, \pm 2$ .

**Remark.** In order to simplify the definition of  $b_{ij}^{\Delta}$  we have cheated a little bit. Whenever the triangle  $\Delta$  is self-folded along an arc  $i$ , then in the right hand side of the definition of the entry  $b_{ij}^{\Delta}$ , the arc  $i$  has to be replaced by its enclosing loop  $l(i)$ , cf. Figure 2.

**Example 2.1.** (1) We compute  $B(T)$  for the triangulated punctured triangle.

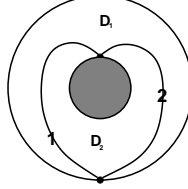


It is:

$$\underbrace{B^{D_1}}_{=0} + B^{D_2} + B^{D_3} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(2) Take an annulus with one marked point on each boundary and the triangulation  $T$  as in the picture. Then  $B(T)$  is

$$B^{D_1} + B^{D_2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$



### 3. CLUSTER ALGEBRAS

In this section we present a very short introduction to cluster algebras, following Fomin-Zelevinsky [FZ02a]. A cluster algebra  $\mathcal{A} = \mathcal{A}(\underline{x}, B)$  is a subring of  $\mathbb{F} = \mathbb{Q}(u_1, \dots, u_m)$ , associated to a *seed*  $(\underline{x}, B)$  defined in the following way.

- (i) A seed is a pair  $(\underline{x}, B)$  consisting of a *cluster*  $\underline{x} = (x_1, \dots, x_m)$  where  $\underline{x}$  is a free generating set of  $\mathbb{F}$  over  $\mathbb{Q}$  and  $B = (b_{xy})_{xy}$  is a sign-symmetric  $m \times m$  matrix with integer coefficients, i.e.  $b_{xy} \in \mathbb{Z}$  for all  $1 \leq x, y \leq m$  and if  $b_{xy} > 0$  then  $b_{yx} < 0$ .
- (ii) A seed  $(\underline{x}, B)$  can be mutated to another seed  $(\underline{x}', B')$ : *mutation at*  $z \in \underline{x}$  is the map  $\mu_z: (\underline{x}, B) \mapsto (\underline{x}', B')$ :  $\underline{x}' = \underline{x} - z \cup z'$  where  $z'$  is defined via the *exchange relation*

$$zz' = \prod_{\substack{x \in \underline{x} \\ b_{xz} > 0}} x^{b_{xz}} + \prod_{\substack{x \in \underline{x} \\ b_{xz} < 0}} x^{-b_{xz}}$$

and  $B'$  is defined similarly via *matrix mutation*:

$$b'_{xy} = \begin{cases} -b_{xy} & \text{if } x = z \text{ or } y = z \\ b_{xy} + \frac{1}{2}(|b_{xz}| \cdot b_{zy} + b_{xz} \cdot |b_{zy}|) & \text{otherwise.} \end{cases}$$

Note:  $\mu$  is involutive, i.e.  $\mu_{z'}(\mu_z((\underline{x}, B))) = (\underline{x}, B)$ .

Two seeds  $(\underline{x}, B)$  and  $(\underline{x}', B')$  are said to be *mutation-equivalent* if one can be obtained from the other through a sequence of mutations.

The *cluster variables* are defined to be the union of all clusters of a mutation-equivalence class (of a given seed). These appear in overlapping sets. Finally, the corresponding *cluster algebra*  $\mathcal{A} = \mathcal{A}(\underline{x}, B)$  is the subring of  $\mathbb{F}$  generated by all the cluster variables. (Here we are defining cluster algebras with trivial coefficients.) A cluster algebra is said to be of *finite type* if there exists only a finite number of cluster variables.

One can show that up to isomorphism of cluster algebras  $\mathcal{A}(\underline{x}, B)$  does not depend on the initial choice of a free generating set  $\underline{x}$ .

**Example 3.1.** (Type  $A_2$ ). We start with the pair  $\underline{x} = (x_1, x_2)$ ,  $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . In a first step we mutate  $x_1$ . from  $x_1 x'_1 = 1 + x_2$  we obtain  $x'_1 = \frac{1+x_2}{x_1}$ . The next mutation is at  $x_2$  (mutation at  $x'_1$  would lead us back to  $x_1$ ), we have  $x'_2 = \frac{x_1+1}{x_2}$ . And then  $x''_1 = \frac{x_1+x_2+1}{x_1 x_2}$ ;  $x''_2 = x_1$ ,  $x'''_1 = x_2$ . In particular, we obtain five cluster variables in this example.

Some of the main results on cluster algebras are summarized here:

- *Laurent phenomenon*:  $\mathcal{A}(\underline{x}, B)$  sits inside  $\mathbb{Z}[x_1^\pm, \dots, x_m^\pm]$ , i.e. every element of the cluster algebra is an integer Laurent polynomial in the variables of  $\underline{x}$  (cf. [FZ02b]);
- Classification of finite type cluster algebras by roots systems, [FZ03b] (cluster algebras of finite type can be classified by Dynkin diagrams);
- Realizations of algebras of regular functions on double Bruhat cells in terms of cluster algebras ([BFZ05]).

Examples of cluster algebras are: Coordinate rings of  $\mathrm{SL}_2$ ,  $\mathrm{SL}_3$  ([FZ04]); Plücker coordinates on  $\mathrm{Gr}_{2,n+3}$  ([Sc], [GSV03]).

**Cluster algebras and quivers.** We will now explain how to associate a quiver to a seed of a cluster algebra.

Recall that a *quiver*  $\Gamma = (\Gamma_0, \Gamma_1)$  is an oriented graph with vertices  $\Gamma_0$  and arrows  $\Gamma_1$  between them. E.g.

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

with  $\Gamma_0 = \{1, 2, 3\}$  and  $\Gamma_1 = \{\alpha, \beta\}$ ;

Any skew-symmetric  $m \times m$ -matrix  $B$  determines a quiver  $\Gamma(B)$  with  $m$  vertices. One labels the columns of  $B$  by  $\{1, 2, \dots, m\}$  and sets  $\Gamma_0 = \{1, 2, \dots, m\}$ . Then one draws  $b_{xy}$  arrows from  $x$  to  $y$  if  $b_{xy} > 0$  (for  $x, y \in \Gamma_0$ ).

Such a quiver has no loops and for any two vertices  $i \neq j$  of  $\Gamma(B)$ , there are only arrows in one direction between them.

So in particular, if the matrix  $B$  of a seed  $(\underline{x}, B)$  is skew-symmetric, it determines a quiver in this way.

**Example 3.2.** The matrix  $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  from Example 3.1 above gives the quiver:

$$1 \longrightarrow 2$$

Clearly, this process is reversible: a quiver whose arrows only go in one direction between any given pair  $i \neq j$  of vertices and without loops gives rise to a skew-symmetric matrix which we will denote by  $B(\Gamma)$ .

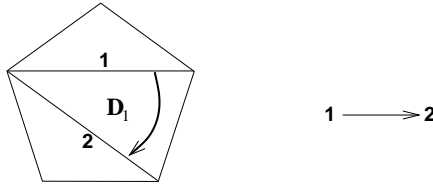
#### 4. FROM TRIANGULATIONS TO CLUSTER ALGEBRAS

From now on we assume that  $(S, M)$  is a disc with at most one puncture. We want to show how a triangulation  $T$  of  $(S, M)$  determines a cluster algebra. Label the arcs of  $T$  by  $1, 2, \dots, n$ .

Then we define a cluster  $\underline{x}_T = (x_1, \dots, x_n)$  by sending  $i \mapsto x_i$  and choose as a matrix the the skew-symmetric matrix associated  $B(T)$  associated to  $T$ . in Section 2. This clearly produces a seed  $(\underline{x}_T, B(T))$ .

Thus to the triangulation  $T$  of the disc  $(S, M)$  we have associated the seed  $(\underline{x}_T, B(T))$  and hence obtain a cluster algebra  $\mathcal{A} = \mathcal{A}(\underline{x}_T, B(T))$ .

**Example 4.1.** Consider an unpunctured pentagon as below. In the triangulation, we label the arcs 1 and 2. They form a triangle  $D_1$  together with a boundary arc and 2 is the clockwise neighbour of 1.



Then the seed we obtain is  $((x_1, x_2), B(T))$  with  $B(T) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  as in Example 3.1 above.

## 5. CLUSTER CATEGORIES

Cluster categories are certain quotients of derived categories of module categories. They were introduced in 2005 by Buan-Marsh-Reineke-Reiten-Todorov ([BMRRT05]).

Independently, Caldero-Chapoton-Schiffler have introduced the cluster categories (in type  $A_n$ ) in 2005 ([CCS06]) using a graphical description. Later, Schiffler extended this to type  $D_n$  in ([Sch06]).

The aim behind the definition of cluster categories was to model cluster algebras using the representation theory of quivers. This was motivated by the observation that the cluster variables of a cluster algebra of finite type are parametrized by the almost positive roots of the corresponding root system.

Cluster categories have led to new development in the theory of the (dual of the) canonical bases, they provide insight into cluster algebras. They have also developed into a field of their own. E.g. they have led to the definition of cluster-tilting theory.

Let us describe the construction of cluster categories, following [BMRRT05].

We start with a quiver  $Q$  whose underlying graph is a simply-laced Dynkin diagram (i.e. of type ADE). Denote by  $D^b(kQ)$  the bounded derived category of finite dimensional  $kQ$ -modules (we assume that the field  $k$  is algebraically closed). Note that the shape of the quiver of  $D^b(kQ)$  is  $Q \times \mathbb{Z}$  with certain connecting arrows. By *quiver of  $D^b(kQ)$*  we mean the Auslander-Reiten quiver of  $D^b(kQ)$ , i.e. the quiver whose vertices are the isomorphism classes of indecomposable modules and whose arrows come from irreducible maps between them.

This quiver has two well-known graph automorphisms:  $\tau$  (“Auslander-Reiten translate”) which sends each vertex to its neighbour to the left. And  $[1]$  (the “shift”) which sends a vertex in a copy of the module category of  $kQ$  to the corresponding vertex in the next copy of the module category.

The *cluster category*,  $\mathcal{C}$ , is now defined as the orbit category of  $D^b(kQ)$  under a canonical automorphism:

$$\mathcal{C} := \mathcal{C}(Q) := D^b(kQ) / \tau^{-1} \circ [1]$$

One can show that this is independent of the chosen orientation of  $Q$ . More generally, Keller ([Ke05]) has introduced the  $m$ -cluster category,  $\mathcal{C}^m$  as follows:

$$\mathcal{C}^m := \mathcal{C}^m(Q) := D^b(kQ) / \tau^{-1} \circ [m]$$

Keller has shown in [Ke05] that  $\mathcal{C}^m$  is triangulated and a Calabi-Yau category of dimension  $m + 1$ . Furthermore,  $\mathcal{C}^m$  is Krull-Schmidt ([BMRRT05]). The  $m$ -cluster category has attracted a lot of interest over the last few years. In particular, it has been studied by Keller-Reiten, Thomas, Wralsen, Zhu, B-Marsh, Assem-Brüstle-Schiffler-Todorov, Amiot, Wralsen, etc.

Our goal for this note is to describe  $\mathcal{C}^m$  using diagonals of a polygon (type  $A_n$ ) and arcs in a punctured polygon (type  $D_n$ ).

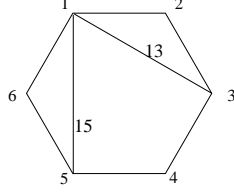
## 6. FROM ARCS VIA QUIVERS TO CLUSTER CATEGORIES

Let us first recall the notion of a stable translation quiver due to Riedtmann [Ri90].

**Definition.** A *stable translation quiver* is a pair  $(\Gamma, \tau)$  where  $\Gamma = (\Gamma_0, \Gamma_1)$  is a quiver (locally finite, without loops) and  $\tau : \Gamma_0 \rightarrow \Gamma_0$  is a bijective map such that

the number of arrows from  $x$  to  $y$  equals the number of arrows from  $\tau y$  to  $x$  for all  $x, y \in \Gamma_0$ . The map  $\tau$  is called the *translation* of  $(\Gamma, \tau)$ .

Now we are ready to define a quiver  $\Gamma$  from a hexagon (see figure below) as follows:

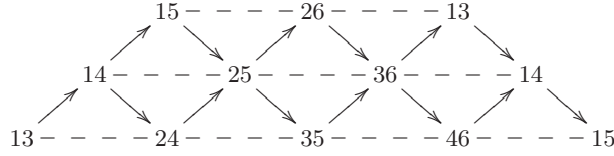


$\Gamma_0$ : The vertices are the diagonals  $(ij)$  of the hexagon ( $1 \leq i < j - 1 \leq 7$ ).

$\Gamma_1$ : The arrows are of the form  $(ij) \rightarrow (i, j + 1)$ ,  $(ij) \rightarrow (i + 1, j)$  provided the target is also a diagonal in the hexagon ( $i, j \in \mathbb{Z}_6$ ).

Set  $\tau: (ij) \mapsto (i - 1, j - 1)$  to be anti-clockwise rotation about the center.

The quiver obtained this way from the hexagon is the following:



It clearly is an example of a stable translation quiver.

Note that such a quiver can be defined for any polygon. Denote the quiver arising in that way by  $\Gamma(n, 1)$  if  $n + 2$  is the number of vertices of the polygon. (The use of  $n$  instead of  $n + 2$  in the notation of the quiver  $\Gamma(n, 1)$  and the extra entry 1 are used to make this compatible with the more general setting involving  $m$ -diagonals described below). Caldero, Chapton and Schiffler have shown that the cluster category can be obtained via diagonals in a polygon:

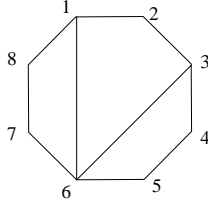
**Theorem 6.1** ([CCS06]). *The quiver of the cluster category  $\mathcal{C} = \mathcal{C}(A_{n-1})$  is isomorphic to the quiver  $\Gamma(n, 1)$  obtained from an  $(n + 2)$ -gon.*

As before in the case of the bounded derived category, the *quiver of  $\mathcal{C}$*  is an abbreviation for the Auslander-Reiten quiver of  $\mathcal{C}$ . It has as vertices the indecomposable objects of  $\mathcal{C}$ , and as arrows are the irreducible maps between them.

To be able to model  $m$ -cluster categories we now generalize the notion of diagonal and introduce the so-called  $m$ -diagonals. We start with a polygon  $\Pi$  with  $nm + 2$  vertices ( $n, m \in \mathbb{N}$ ), labeled by  $1, 2, \dots, nm + 2$ .

**Definition.** An  $m$ -diagonal is a diagonal  $(ij)$  dividing  $\Pi$  into an  $mj + 2$ -gon and an  $m(n - j) + 2$ -gon ( $1 \leq j \leq \lceil \frac{n-1}{2} \rceil$ ).

**Example 6.2.** To illustrate this, let  $\Pi$  be an octagon,  $n = 3$ ,  $m = 2$ . In that case,  $1 \leq j \leq 1$ , so any 2-diagonal has to divide  $\Pi$  into a quadrilateral and a hexagon.



Observe that each maximal set of non-crossing 2-diagonals contains two elements. They are  $\{(16), (36)\}$ ,  $\{(16), (25)\}$ ,  $\{(16), (14)\}$  and rotated version of these.



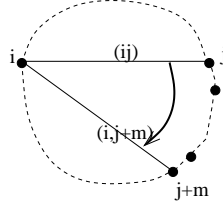
Recall that the number of arcs in a triangulation (see Section 2) is an invariant of a disc  $(S, M)$ , called the rank of  $(S, M)$ . In the same way, the maximal number of non-crossing  $m$ -diagonals is an invariant of the polygon. It is equal to  $n - 1$  (for the  $nm + 2$ -gon  $\Pi$ ).

Using  $m$ -diagonals we can now define a translation quiver  $\Gamma(n, m) = (\Gamma, \tau_m)$ :

$\Gamma_1$ :  $(ij) \rightarrow (ij')$  if  $(ij)$ ,  $B_{jj'}$  and  $(ij')$  span an  $m + 2$ -gon ( $B_{jj'}$  is the boundary  $j$  to  $j'$ , going clockwise).

$\Gamma_0$ : The vertices are the  $m$ -diagonals  $(ij)$  in  $\Pi$  (with  $1 \leq i < j - m$ ).

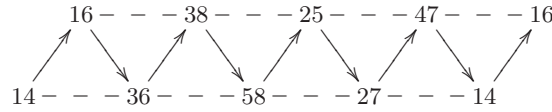
$\Gamma_1$ : The arrows are of the form  $(ij) \rightarrow (i, j + m)$ ,  $(ij) \rightarrow (i + m, j)$  provided the target is still inside the polygon. In other words:  $(ij)$ ,  $(i, j + m)$  and the boundary arc  $j$  to  $j + m$  (resp.  $(ij)$ ,  $(i + m, j)$  and the boundary arc from  $i$  to  $i + m$ ) form an  $m + 2$ -gon as in the picture:



Furthermore, let  $\tau_m$  be anti-clockwise rotation (about center) through the angle  $m \frac{2\pi}{nm+2}$ .

**Remark.** It is clear that  $\Gamma(n, m)$  is a stable translation quiver. In case  $m = 1$ , we recover the usual diagonals.

The quiver  $\Gamma(3, 2)$  for the octagon from the previous example is thus:



Then one can show that the  $A$ -type  $m$ -cluster category can be obtained using  $m$ -diagonals in a polygon:

**Theorem 6.3** ([BM08a]). *The quiver of the  $m$ -cluster category  $\mathcal{C}^m = \mathcal{C}^m(A_{n-1})$  is isomorphic to the quiver  $(\Gamma(n, m), \tau_m)$  obtained from  $m$ -diagonals in an  $nm + 2$ -gon.*

The proof of our result uses Happel's description of the Auslander-Reiten-quiver of  $D^b(kQ)$  where  $Q$  is of Dynkin type  $A_{n-1}$  and combinatorial analysis of  $\Gamma(n, m)$ . For details we refer to [BM08a, Section 5].

**The description in type  $D$ .** We have a similar description of the  $m$ -cluster categories of  $D$ -type. Instead of working with a polygon (or unpunctured disc) we now have to use a punctured polygon. Let  $\Pi$  be a punctured  $nm - m + 1$ -gon. Instead of using the term diagonal, we now speak of *arcs* in  $\Pi$ . An arc going from  $i$  to  $j$ , homotopic equivalent to the boundary  $B_{ij}$  from  $i$  to  $j$  (going clockwise) is denoted by  $(ij)$ . By  $(ii)$  we denote an arc homotopic equivalent to the boundary  $B_{ii}$  with endpoints in  $i$ . And  $(i0)$  is an arc homotopic equivalent to the arc between  $i$  and the puncture 0. We will say that an  $n$ -gon is *degenerate* if it has  $n$  sides and  $n - 1$  vertices.

For details and examples we refer to [BM07, Section 3].

**Definition.** An  $m$ -arc of  $\Pi$  is an arc  $(ij)$  such that

- (i)  $(ij)$  and  $B_{ij}$  (the boundary from  $i$  to  $j$ , going clockwise) form an  $km + 2$ -gon for some  $k$ ,
- (ii)  $(ij)$  and  $B_{ji}$  (the boundary from  $j$  to  $i$ , going clockwise) form an  $lm + 2$ -gon for some  $l$ .

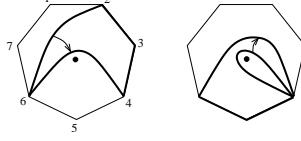
Furthermore,  $(ii)$  and  $(i0)$  are called  $m$ -arcs if  $(ii)$  and  $B_{ii}$  form a degenerate  $km + 2$ -gon for some  $k$ .

Then we can define a translation quiver  $\Gamma = \Gamma_{\odot}(n, m)$  as follows:

$\Gamma_0$ : The vertices are the  $m$ -arcs of  $\Pi$

$\Gamma_1$ : The arrows are the so-called  $m$ -moves between vertices:

We say that  $(ij) \rightarrow (ik)$  is an  $m$ -move if  $(ij)$ ,  $B_{jk}$  and  $(ik)$  span a (degenerate)  $m + 2$ -gon. In the figure below there are two examples of 2-moves.



$\tau_m$ : rotation anti-clockwise (about center), through an angle of  $\frac{m}{nm-m+1}$ .

Clearly, the pair  $(\Gamma_{\odot}(n, m), \tau_m)$  is also a stable translation quiver. We can now formulate the statement.

**Theorem 6.4** (Theorem 3.6 of [BM07]). *The quiver of the  $m$ -cluster category  $\mathcal{C}^m = \mathcal{C}^m(D_n)$  is isomorphic to the quiver  $(\Gamma_{\odot}(n, m), \tau_m)$  obtained from  $m$ -arcs in an  $nm - m + 1$ -gon.*

**The  $m$ -th power of a translation quiver.** We will now describe another way to obtain  $m$ -cluster categories directly from the diagonals or arcs in a (punctured) polygon. Let  $(\Gamma, \tau)$  a translation quiver as before. Then we define the  $m$ -th power of  $\Gamma$ ,  $\Gamma^m$ , to be the quiver whose vertices are the vertices of  $\Gamma$  (i.e.  $\Gamma_0^m = \Gamma_0$ ). The arrows in  $\Gamma^m$  are paths of length  $m$ , going in an unique direction. (To be precise, we ask that such a path is *sectional*, i.e. that in a path  $x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_{m-1} \rightarrow x_m$  of length  $m$  we have  $\tau x_{i+1} \neq x_{i-1}$  whenever  $\tau x_{i+1}$  is defined.) And the translation  $\tau^m$  of  $\Gamma^m$  is obtained by repeating the original translation  $m$  times.

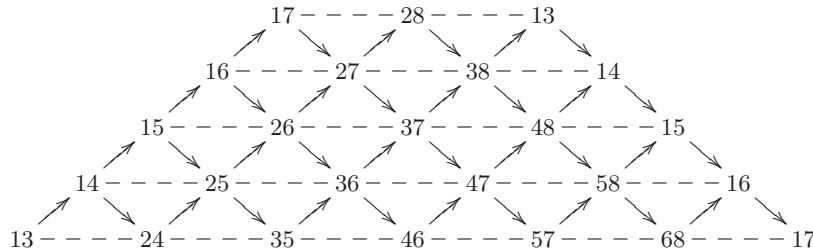
**Definition.** The quiver  $(\Gamma^m, \tau^m)$  as defined above is called the  $m$ -th power of  $(\Gamma, \tau)$ .

With this we are ready to formulate the result:

**Theorem 6.5** ([BM08a]).  $\Gamma(n, m)$  is a connected component of  $(\Gamma(nm, 1))^m = (\Gamma(\text{cluster category}))^m$ .

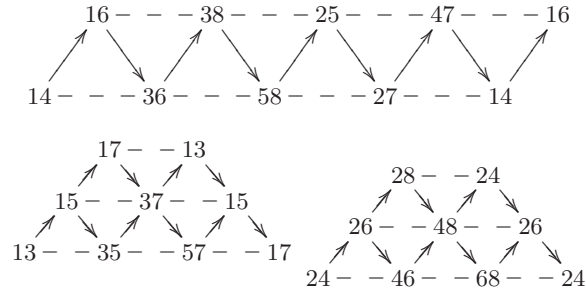
**Remark.** Observe that  $(\Gamma^m, \tau^m)$  is again a stable translation quiver. However, even if  $(\Gamma, \tau)$  is connected, the  $m$ -th power is not connected in general!

**Example 6.6.** To illustrate this consider the quiver  $\Gamma(6, 1)$  of an octagon.



Its second power has three components: one component is  $\Gamma(2, 2)$ . The two other components are both quivers of quotients  $D^b(A_3)/[1]$  of  $D^b(A_3)$  by the shift. In particular, we have thus given a geometric construction of a quotient of  $D^b(A_3)$  which is not an  $m$ -cluster category!

The three components of the second power of  $\Gamma(6, 1)$  are:



**Remark.** We have a corresponding result for type  $D$ , see Theorem 5.1 of [BM07]. In addition, in type  $D$  we give an explicit description of *all* connected components appearing in the  $m$ -th power of  $\Gamma_{\odot}(nm - m + 1, 1)$

## 7. CONNECTIONS AND FUTURE DIRECTIONS

To finish we want to provide a short outlook and describe some open problems and possible future directions.

- In recent work with Robert Marsh ([BM08b]) we provide a link between cluster algebra combinatorics and perfect matchings (for vertices and edges of a triangulation). This uses work of Conway-Coxeter ([CC73a], [CC73b]), and of Broline-Crowe-Isaacs ([BCI74]) on frieze patterns of positive integers.
- $Y$ -systems can be defined in general for a pair  $(G, H)$  of Dynkin types. Zamolodchikovs periodicity conjecture for general  $Y$ -systems have been proved for  $G = A_1$  and  $H = A_n$  by Frenkel-Szenes ([FS95]), by Gliozzi-Tateo ([GT96]) and for  $G = A_1$ ,  $H$  any Dynkin type by Fomin-Zelevinsky ([FZ03a]) using cluster algebras theory. More recently, the cases  $G = A_k$ ,  $H = A_n$  have been solved ([Sz06], [Vo06], independently). In this context, there are various open questions. First of all: can periodicity be proved for  $G$  of arbitrary of Dynkin type using the theory of cluster algebras? This is not even known for  $G = A_k$ . Second: what would be a good counterpart on the side of  $Y$ -systems to the geometric model for  $m$ -cluster categories? And thirdly: In current work with Marsh we have discovered classes of infinite periodic systems ( $G = A_1$ ,  $H = A_{\infty}$ ). Does this have a translation to the setting of  $Y$ -systems?
- The approach to model cluster algebras with discs  $(S, M)$  works for types  $A$  and  $D$  ([FST06]) and for types  $B$ ,  $C$  under certain modifications ([CFZ02]). *Open:* what can be said about the exceptional types, in particular, is there a way to model type  $E$  using a disc with marked points?
- Jorgensen ([Jo07]) has obtained  $m$ -cluster categories as quotient categories of cluster categories via deletion of rows ( $\tau$ -orbits). They inherit a triangulated structure. This process can be viewed as a reverse to our construction using the  $m$ -th power of a quiver. *Question:* how can we explain the triangulated structure of  $m$ -cluster categories via the  $m$ -th power of a quiver?

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